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RESOLVING THE INNERMOST REGION OF THE ACCRETION DISK OF THE LENSED QUASAR Q2237+0305 THROUGH GRAVITATIONAL MICROLENSING

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ABSTRACT

We study three high magnification microlensing events, generally recognized as probable caustic crossings, in the optical light curves of the multiply imaged quasar Q2237+0305. We model the light curve of each event as the convolution of a standard thin disk luminosity profile with a straight fold caustic. We also allow for a linear gradient that can account for an additional varying background effect of microlensing. This model not only matches noticeably well the global shape of each of the three independent microlensing events but also gives remarkably similar estimates for the disk size parameter. The measured average half-light radius, $R_{1/2} = (3.0 \pm 1.5)\sqrt{M/0.3M_\odot}$ light-days, agrees with previous estimates. In the three events, the core of the magnification profile exhibits "fine structure" related to the innermost region of the accretion disk (located at a radial distance of 2.7 ± 1.4 Schwarzschild radii according to our measurement). Relativistic beaming at the internal rim of the accretion disk can explain the shape and size of the fine structure, although alternative explanations are also possible. This is the first direct measurement of the size of a structure, likely the innermost stable circular orbit, at ∼3 Schwarzschild radii in a quasar accretion disk. The monitoring of thousands of lensed quasars with future telescopes will allow the study of the event horizon environment of black holes in hundreds of quasars in a wide range of redshifts ($0.5 < z < 5$).

Key words: accretion, accretion disks – gravitational lensing: micro – quasars: individual (Q2237+0305)

1. INTRODUCTION

The structure of the central engine of quasars can potentially be resolved right to the supermassive black hole event horizon by microlensing caustic crossing, a phenomenon related to gravitational lensing of distant quasars by intervening lens galaxies (Chang & Refsdal 1979, 1984; Wambsganss 2006). The main contributor to the gravitational field of the lens galaxy is the smoothly distributed dark matter. But there are also compact sources (stars and their remnants) that locally break the smoothness of the gravitational field and split each image of the quasar in several micro images. When, due to the movement relative to the galaxy, a source changes its position relative to the distribution of stars, the number of micro images can change. Regions where the source is mapped to a different number of images are separated by caustic curves where the lens mapping bifurcates and the magnification is formally infinity. When a sufficiently small sized source crosses a caustic its magnification rises with a very large gradient at the caustic and then decreases following a $D^{-1/2}$ dependence with the distance to the caustic (straight fold caustic approximation; Schneider & Weiss 1987). In this case, the convolution of the source luminosity and the caustic magnification profiles can keep a great deal of information about the structure of the source. When the size of the source is relatively large, however, the phenomenology of caustic crossing becomes more complex and the source structure can be hard to deconvolve (Shalyapin et al. 2002; Mosquera et al. 2009).

Q2237+0305 is a particularly interesting system for studying caustic crossings. The unusually low redshift of the lens galaxy allows the quasar images to be seen through the galaxy bulge in a region of high density of stars (and hence of caustics). The relatively high rate of caustic crossings is also favored by the high relative velocities between the source and the lens galaxy. Our aim is to use the caustic crossings observed in the optical light curves available for Q2237+0305 to study the source luminosity profile. In particular, we would like to test the classical thin disk model, infer the disk size, and set the spatial scale of the possible substructure of the source. Several of these objectives have, in fact, been attempted in an ample number of works with heterogeneous methodology and results. A group of authors (Shalyapin et al. 2002; Goicoechea et al. 2003; Koptelova et al. 2007; Abolmasov & Shakura 2012) has focused on the region around the high magnification event (HME) peak. This approach is prone to give small estimates of the disk size. Another group of authors has attempted the fit of the light curve in a region around the event sufficiently large (some authors considered even the fit of the whole light curves) as to constrain the baseline far enough from the caustic crossing event, and also to account for the intrinsic source variability (Kochanek 2004; Anguita et al. 2008; Eigenbrod et al. 2008a, 2008b; Poindexter & Kochanek 2010; Sluse et al. 2011). The estimates of the different authors for the size of the source exhibit a large scatter (between 0.2 and 10 light-days) obviously affected by the difference in methodology but also, in many cases, by the lack of information about the effective transverse velocity between the source and the galaxy (see, e.g., Table 4 of Sluse et al. 2011).

Very recently we have obtained, from the statistics of the number of caustic crossings, a measurement of the transverse velocity in Q2237+0305 (Mediavilla et al. 2015) independent of the size of the source that will allow us to map the time scale of the light curves to lengths. In Mediavilla et al. (2015) a preliminary comparative study of the shape of the microlensing events was done to obtain a rough estimate of the size of the events from the FWHM of their average profile. However, no direct attempt to fit the luminosity profile of the source was made. Our approach in this paper, then, is to analyze the three events generally recognized as caustic crossings, removing intrinsic variability. In order to do so, we consider a large enough region to constrain the baseline of each caustic crossing event. Finally, we use our independent estimate of the velocity to set the spatial scale at the quasar.

The phenomenology of these three events is not so simple as to allow a good fitting using the straight fold caustic model alone. A more comprehensive description of microlensing effects based on the numerical simulation of microlensing light curves from magnification maps has been reasonably successful in analyzing the combined influence of microlensing and intrinsic variabillty in several images at once (Kochanek 2004; Poindexter & Kochanek 2010). However, this numerical approach is a computational challenge when detailed source models with several parameters (inner and outer radii, scale parameter, inclination, orientation respect to the caustic, slope of the luminosity profile, etc.) are to be considered. Thus, we would like to identify the simplest microlensing phenomenology beyond the straight fold caustic crossing that can be incorporated into this approximation to obtain good analytical fits. If the complexity of the events arises from the combination of the effects of more than one caustic crossing or to large effects of caustic curvature, the attempt will be very difficult. On the contrary, if the additional complexity corresponds to a background gradient in the magnification, we can try to account for this effect by including an ad hoc extra term, linear in the distance to the caustic. We will show that this last possibility works very well in the case of the three events selected as the best candidates for single caustic crossing.

Thus, in Section 2 we present and discuss the results of the analysis of the three events under the straight fold caustic approximation with an additional linear term. In Section 3 we fit a model including relativistic effects to the fine structure detected in the luminosity profiles and determine its spatial scale. Finally, in Section 4 the main results are summarized.

2. ANALYTICAL FITTING OF THREE HMEs

We start by assuming that an HME can be modeled by a single caustic crossing plus a linearly varying background effect of microlensing. To remove the intrinsic variability of the source we divide the light curve with the HME by the light curve of the image with the smoothest variations in the epoch of interest.[8] This last image could be also affected by microlensing but we will consider that this variation is also included in the linear gradient term.

Thus, we fit each HME with a simple model resulting from the convolution of a straight fold caustic magnification profile (Schneider & Weiss 1987; Shalyapin et al. 2002) with a thin disk luminosity profile (Shakura & Sunyaev 1973) plus a linear term:

$$f = c_0 + c_J J\left((t - t_0)/\Delta t\right) + c_1 t, \quad (1)$$

with,

$$J\left((t - t_0)/\Delta t\right) = \frac{1}{\pi\Gamma\left[\frac{11}{3}\right]Z\left[\frac{8}{3}\right]}\int_0^{+\infty} dx x^{-1/2}\int_{-\infty}^{+\infty} dy \times\left[\exp\left(\left(\sqrt{\left(x - (t - t_0)/\Delta t\right)^2 + y^2}\right)^{4/3}\right) - 1\right]^{-1}, \quad (2)$$

where the relevant physical parameters are the disk scale parameter expressed in days, $\Delta t = R_\lambda/v_{\rm eff}$, and the caustic crossing time, t_0. R_λ is the disk scale parameter and $v_{\rm eff}$ the relative velocity between the caustic and the source. This model is linear in c_0, c_J and c_1 but nonlinear in t_0 and Δt.

We fit the model described in Equation (1) to the three HME generally classified as probable single caustic crossings: two events in image A at JD (245)1500 (Wyithe et al. 2000; Shalyapin 2001; Goicoechea et al. 2003; Kochanek 2004; Gil-Merino et al. 2006; Koptelova et al. 2007; Abolmasov & Shakura 2012) and at JD 4000 (Eigenbrod et al. 2008a, 2008b), and one event in image C at JD 1360 (Wyithe et al. 2000; Shalyapin 2001; Kochanek 2004; Koptelova et al. 2007; Anguita et al. 2008; Abolmasov & Shakura 2012). We have used the Optical Gravitational Lensing Experiment (OGLE, Woźniak et al. 2000; Udalski et al. 2006) optical light-curves of Q2237+0305 joined, in the case of the A image event at JD 1500, with the Gravitational Lensing International Time Project (GLITP) light-curve (Alcalde et al. 2002). We have selected a large time interval (~800 days) around the peak of each HME to properly define the caustic baseline, a very important step to perform the comparison of the events. To remove intrinsic variability we have divided the light curve of each event by the light curve of the image that exhibits less variability during the event (image B for the events A/JD 1500 and C/JD 1360 and image D for the event A/JD 4000).

The model succeeds in fitting the global structure of the three caustic crossings (Figure 1) with a remarkably good agreement between the scale parameters (Table 1). To estimate the errors in the fluxes (see parameter $\langle\sigma_{\rm adj}\rangle$ in Table 1) including all the sources of uncertainty we have computed the average of the absolute value of the differences between points adjacent in the light curves separated by 2 days or less (a very conservative estimate of the error). In Figure 2 we have over-plotted the three events after subtraction of the linear component, re-centering, and normalization in both amplitude and scale parameter. There is an outstanding global coincidence between the profiles very difficult to explain without accepting the hypothesis of a shared common phenomenology. Perhaps the coincidence is not so surprising taking into account that the three HME have, in fact, been selected among others as the best candidates for single caustic crossing.

The average of the scale parameter expressed in days is $\Delta t = (R_\lambda/v_{\rm eff}) = 56.5$ days, and its dispersion, $\sigma_{\Delta t} = 4.4$ days, corresponding to a half-light radius of $R_{1/2} = 2.44R_\lambda = 137.86 \pm 10.74$ days. To map this time interval in the light curve into a length at the quasar plane we will use the measurement of the source velocity in Q2237+0305 by Mediavilla et al. (2015), $v_{\rm eff} = (493 \pm 246)\sqrt{M/0.17M_\odot}$ km s^{-1}, consistent with previous upper (Wyithe et al. 1999; Gil-Merino et al. 2005) and lower (Poindexter & Kochanek 2010) limits, obtaining, $R_{1/2} = (3.0 \pm 1.5)$

[8] Notice that the time delay between images is small enough to be neglected.

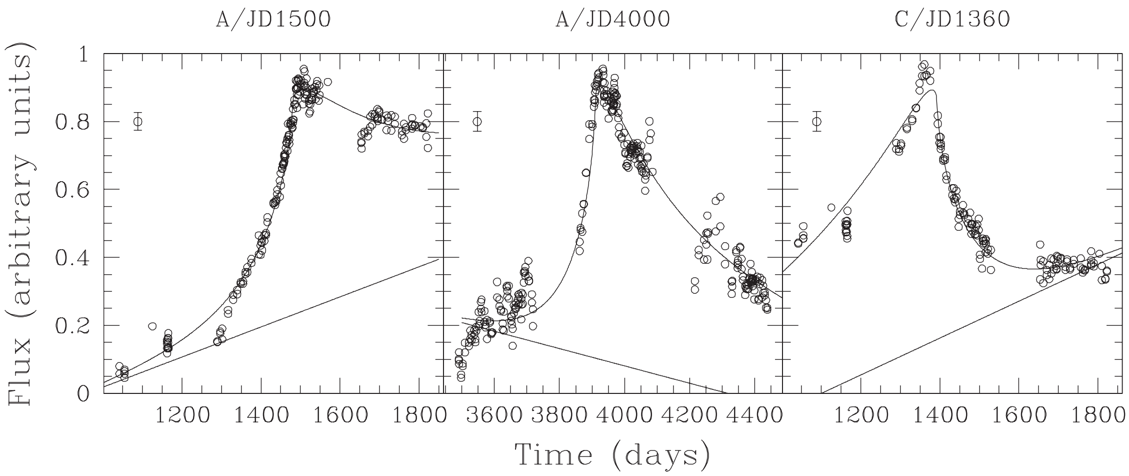

Figure 1. Independent fits for each of the three HMEs. Points correspond to the light curves. Continuous curves are the fits to the model obtained from the convolution of a thin disk profile with a straight fold caustic plus a linear term. Errors bars (see text) are included at the upper left corners.

Table 1
Fit Parameters for the Three HMEs

Event	Δt (days)	$\langle \sigma_{\mathrm{adj}} \rangle$	χ^2_{red}
A/JD 1500	51.5^{+3}_{-3}	0.06	1.52
A/JD 4000	60.0^{+9}_{-8}	0.02	3.6
C/JD 1360	58.0^{+5}_{-5}	0.05	1.82

$\sqrt{M/0.3M_\odot}$ light-days at $\lambda_{\mathrm{rest}} = 2018$ Å (notice that the main source of uncertainty is v_{eff}). This value is in agreement with previous determinations (see Mediavilla et al. 2015; Muñoz et al. 2015 and references therein).

3. RESOLVING THE INNERMOST REGION OF THE ACCRETION DISK FROM THE FINE STRUCTURE OF THE MAGNIFICATION PROFILES

As commented in Section 1, the possibility of resolving the inner structure of the quasar accretion disk just to a few gravitational radii ($R_g = GM/c^2$) is perhaps the most challenging motivation for caustic crossing studies. Among the three HME that we are studying, the event at A/JD 1500 is the one with the best available light curves. In 1999 and 2000 we conducted a daily photometric monitoring (GLITP; Alcalde et al. 2002) of Q2237+0305 with the Nordic Optical Telescope that allowed us to map with good sampling and accuracy the peak of this HME. Its light curve does not have a smooth shape at the maximum but shows, in both R and V bands, fine structure in the form of a narrow peak followed by a dip and a possible smooth secondary peak (two-peaked structure), qualitatively interpreted by Abolmasov & Shakura (2012) as a signature of the inner disk structure. Abolmasov & Shakura fit the two-peaked structure with a high inclination Kerr model that, in principle, could help to estimate sizes in the source, but the required values for the black hole mass and the effective velocity of the source were so different from expectations that no consistent estimate of the size of the substructure can be based on this model alone.

We will use the independent (not based on light curve fitting) measurement of v_{eff} by Mediavilla et al. (2015) as an additional constraint to infer the substructure size from the three HME light curves. To do that the first step is to look for fine structure (consistent with that found in the event at A/JD 1500) around the maximum in the events at A/JD 4000 and C/JD 1360 that could confirm or deny that the inner region of the accretion disk is resolved in the three caustic crossing magnification profiles.

As seen in Section 2 the global shape of the three HME can be consistently fitted by the caustic crossing model (see Figure 2). However, there are some features of the HME profiles that the model is unable to fit. Some of them, like the ones around 400 days or −200 days in Figure 2 are likely related to the complexity of the HME beyond the straight caustic crossing approximation. Around the maximum of the HME, however, the inability of the model to fit the data is more likely related to the presence of substructure in the source. If the substructure is real, evidence of it should appear in the three events. To ascertain this, we can look at a zoom of the overplotted profiles in the region around the maximum (see inset in Figure 2). We see that the two-peaked structure first detected in the A/JD 1500 event is also present in the other HME of image A at JD 4000. The event in image C also matches very well the highest peak just to the central dip but it does not reach the secondary maximum.[9]

To study the significance of the substructure we have averaged the data in 5-day intervals obtaining the mean profile plotted in Figure 3. Taking the standard errors in the mean inferred from the averaging as reference, we can conclude the existence of fine structure in the core of the HME, with a narrow outstanding peak and a probable secondary smooth maximum. As commented above, to measure the size of the fine structure, we use the estimate of v_{eff} in Q2237+0305 by Mediavilla et al. (2015). We find that the time separation between the peaks of the fine structure (about 50 days) corresponds to a distance of 0.8 ± 0.4 light-days at the quasar that for a black hole mass of $1.2 \times 10^9 M_\odot$ (Assef et al. 2011) corresponds to 5.3 ± 2.7 Schwarzshild radii, i.e., comparable to the diameter of the innermost stable circular orbit (ISCO) in Schwarzschild space. Alternatively, taking as baseline the dip between peaks, the primary peak has a FWHM of 25 days, approximately, that corresponds to 2.7 ± 1.4 Schwarzshild radii. These are strong quantitative evidence that the

[9] Although this difference could be explained by noise (the light curve of image C has only a couple of points in the region of the secondary peak) the convolution of a Kerr model of high inclination with caustics at different position angles can qualitatively explain the existence of either one or two narrow peaks in the magnification profile (Abolmasov & Shakura 2012).

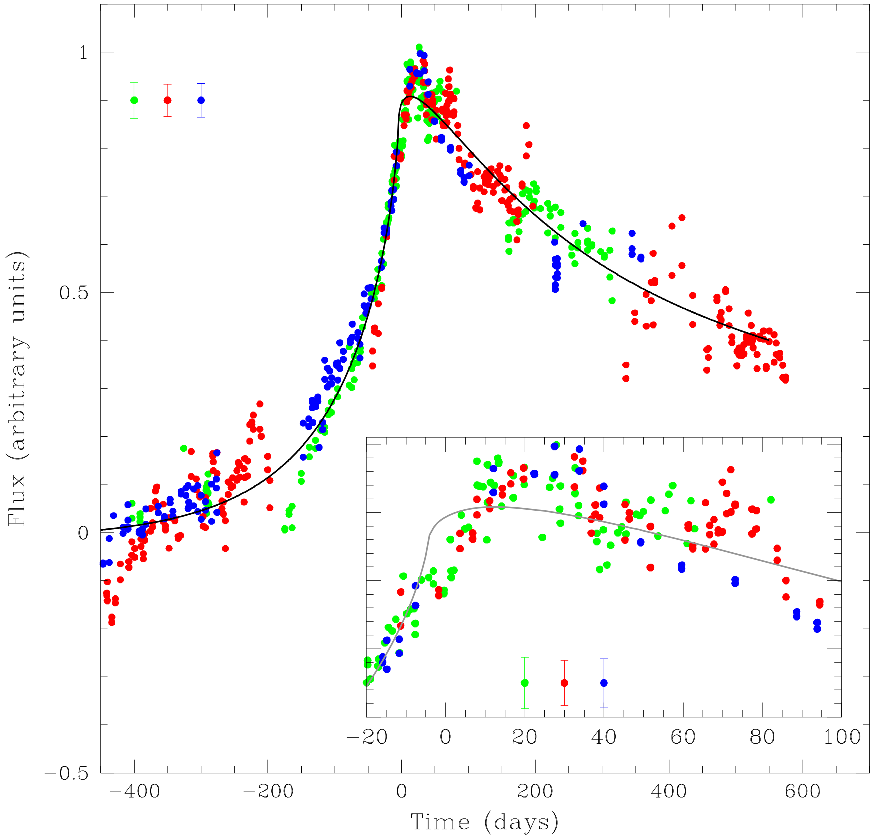

Figure 2. Global comparison of the three HMEs and analytical fit to a single caustic crossing. The inset is a zoom in the region of the maximum. Dots correspond to the experimental light curves of each of the three events (green: A/JD 1500, red: A/JD 4000, blue: C/JD 1360), after removing the linear terms of the fits (see Figure 1), re-centering, and normalization. The continuous curve is the fit of a model without linear term to the whole set of data. Average error estimates (see the text) for each of the HME data are included.

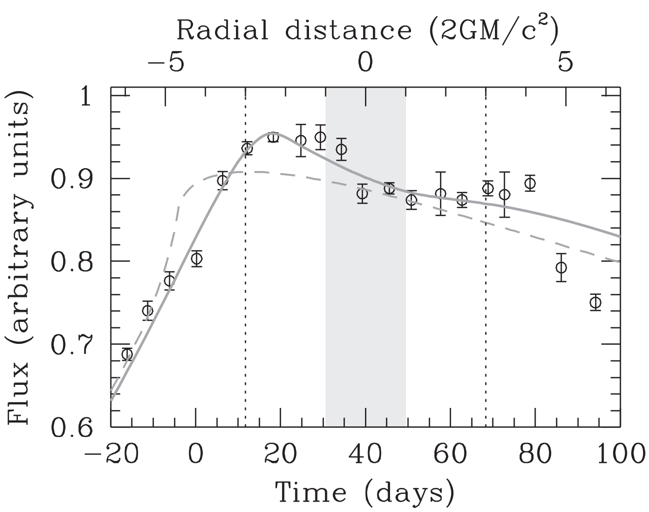

Figure 3. Average of the three HMEs of microlensing in the region of the maximum. Points correspond to the average in intervals of 5 days. Error bars are the standard errors in the mean corresponding to each 5-day interval. The continuous curve corresponds to a relativistic model with innermost stable orbit at 3 Schwarzschild radii. The dashed curve is the non-relativistic fit of Figure 2. Dotted vertical lines correspond to ±3 Schwarzschild radii (ISCO) and the shaded region to ±1 Schwarzschild radii (event horizon).

observations of the three selected HME of Q2237+0305 are resolving the innermost structure of the quasar accretion disk. The size of the inner disk radius is also consistent with the results by Reynolds et al. (2014).

A non-relativistic thin disk model with an inner hole limited by the ISCO may induce fine structure at the maximum but would make less prominent the peak closer to the sharp side of the caustic, contrary to the observations. It has been proposed that differential relativistic beaming of the approaching and receding sides of the inner rim of the disk can explain the relative intensity of the peaks (Abolmasov & Shakura 2012). We explore this possibility introducing relativistic beaming[10] in the thin disk model and considering the convolution of an inclined disk with the caustic but using our new measurements of the disk parameter scale and of the size of the fine structure. Fixing the scale parameter to the average of the values in Table 1 ($\Delta t = 56.5$ days) and setting the ISCO radius to 3 Schwarzschild radii (in agreement with our estimate of 2.7 ± 1.4 Schwarzschild radii) we have fitted the relativistic model to the innermost region obtaining the fit shown in Figure 3. According to this fit, a model with the ISCO at 3 Schwarzschild radii can consistently account for the shape of the fine structure when high inclination, $i \sim 73°$, is allowed, which can be controversial (Poindexter & Kochanek 2010; Abolmasov & Shakura 2012). The inclusion of relativistic beaming has a negative impact in the fit of the wings of the HME even when changes in the scale parameter or in the slope of the thin disk luminosity profile are allowed. It is probable that these problems can be addressed by modifying the model

[10] At each point of the disk, we correct the flux considering the specific intensity invariant, I_ν/ν^3, and the Doppler and gravitational shifts of the frequency, ν.

and, in any case, the picture that emerges from the relativistic fitting is indeed very suggestive, but alternative possibilities such as the presence of an inhomogeneity in the disk (bright spot) close to the ISCO should also be considered (see, e.g., Moriyama & Mineshige 2015).

4. CONCLUSIONS

The inclusion of a linear term in the analytical model of straight fold caustic crossing by a thin disk, has allowed us to obtain very good fits to the global shape of three selected HME in the light curves of the lensed quasar Q2237+0305. We found excellent agreement between the independent source half-light radii estimates of the three fits and an outstanding coincidence between the experimental profiles of the three HME after removing the linear term (in support of the adopted hypothesis of straight fold caustic crossing). The most important results of this study are:

1. We determine an average half-light radius for the source of $R_{1/2} = (3.0 \pm 1.5)\sqrt{M/0.3M_\odot}$ light-days at $\lambda_{\rm rest} = 2018$ Å in agreement with other results (see, e.g., Muñoz et al. 2015 and references therein).
2. We detect a two-peaked structure in the A/JD 4000 HME that matches reasonably well with the analogous structure previously found in the A/JD 1500 HME (see Abolmasov & Shakura 2012) and partially with the narrow peak observed in the C/JD 1360 HME. This coincidence supports the detection of fine structure in the HME profiles that can be related to the innermost regions of the accretion disk.
3. We set the spatial scale in the HME profiles measuring a separation between the peaks of the substructure of about 0.8 ± 0.4 light-days equivalent to 5.3 ± 2.7 Schwarzschild radii for a $1.2 \times 10^9 M_\odot$ black hole. This is quantitative evidence that the innermost region of the accretion disk, at the ISCO for a Schwarzshild metric, is resolved in the HME of Q2237+0305. In fact, relativistic effects, like beaming acting on an inclined disk, may explain the intensity ratio between the substructure peaks (see also Abolmasov & Shakura 2012).

Taking into account the results shown here and the high observational expectations that some relatively easy experimental improvements can bring (large telescope, adaptative optics, multi-band photometry, continued daily and occasionally hourly observations, etc.), the monitoring of Q2237+0305 should be resumed as a unique source of information about the black hole environment in quasars. In the future, ten years of monitoring of thousands of lensed quasars with the Large Synoptic Survey Telescope (Marshall et al. 2010), will provide hundreds of HME (assuming an average Einstein crossing timescale of 20 years (Mosquera & Kochanek 2011) and 1 caustic per crossing) scanning the innermost environment of supermassive black holes in a sample of quasars with a large range in redshifts, $0.5 < z < 5$ (Mosquera & Kochanek 2011), greatly improving statistical and evolutive studies of supermassive black holes and accretion disks in quasars.

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